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Duality theorems and algorithms for linear programming in measure spaces

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Abstract. The purpose of this paper is to present some results on linear programming in measure spaces (LPM). We prove that, under certain conditions, the optimal value of an LPM is equal to the optimal value of the dual problem (DLPM). We also present two algorithms for solving various LPM problems and prove the convergence properties of these algorithms.

Key words: algorithms, duality theory, Linear programming in measure spaces

1. Introduction

Let (E,F) and (Z,W) be two dual pairs of ordered vector spaces. Let E_+ and Z_+ be the positive cones for E and Z, respectively; and E_+^* and Z_+^* be the polar cones of E_+ and Z_+, respectively. Given $b^* \in F$, $c \in Z$, and a linear map $A: E \to Z$, then the linear programming problem and its dual problem can be formulated as follows:

$$\begin{array}{lll} \text{(LP):} & \text{minimize} & \langle x, b^* \rangle \\ & \text{subject to} & Ax - c \in Z_+ \text{ and } x \in E_+; \end{array}$$

$$\begin{array}{lll} \text{(DLP):} & \text{maximize} & \langle c, y^* \rangle \\ & \text{subject to} & b^* - A^* y^* \in E_+^* \text{ and } y^* \in Z_+^*, \end{array}$$

where A^* represents the adjoint mapping of A. If we assume that A is continuous for the weak topologies on E and Z (w.r.t. F and W, resp.), then (DLP) is the dual problem of (LP) in Kretschmer's sense ([7]).

We now discuss this kind of linear programming in measure spaces. In [2], Glashoff and Gustafson (1982) discussed linear semi–infinite programming (LSIP) in which $E = F := \mathbb{R}^n$, $Z = W := \mathbb{R}^T$ (the linear space of real valued functions on T which vanish everywhere except on a finite subset), $E_+ := \mathbb{R}_+^n, Z_+ := \mathbb{R}_+^T$. The theory and algorithms for linear semi–infinite programming are discussed in [2, 4, 12, 15]. A great deal of theory and algorithms can be also found in Goberna and López (1998) [3]. The generalized capacity problem is an infinite dimensional linear programming problem, which extends the linear semi-infinite programming problem from the variable space $\mathbb{R}^n$ to variable space of the regular Borel measure space. In [8] Lai and Wu (1992) investigate the generalized capacity problem (GCAP), in which E and W are both spaces of regular Borel measures and F and Z are both spaces of continuous real valued functions. Kellerer (1988) [6] explores linear programming in measure spaces by using a theoretical model. He considers linear programming problems for measure spaces of the form:

(P′):minimize $\int_X h d\mu$
subject to $\mu p_* \geqslant \nu$ in $M(Y)$,
where p_* maps $\mu \in M^+(X)$ to $M(Y)$,

and

(D′):maximize $\int_Y g d\nu$
subject to all measurable functions $g \geqslant 0$ on Y
and $pg \leqslant h$, where p maps the set of nonnegative
measurable functions on Y to measurable functions
on X.

Here X and Y are topological spaces endowed with their Borel σ-algebras, and $M^+(X)$ denotes the set of nonnegative measures in $M(X)$. Lai and Wu (1994) [10] discuss LPM and DLPM (defined in the next section) with constraint inequalities on the relationships of measures to measures. They prove that under certain conditions the LPM problem can be reformulated as a general capacity problem as well as a linear semi–infinite programming problem. [10] developes a special type of algorithm when certain conditions are added to an LPM. In the present paper, we develop the dual problem DLPM for LPM and discuss the relationship between the optimal value V(LPM) of an LPM and the optimal value V(DLPM) of a DLPM. In Section 3, we develop algorithms for different types of LPM problems. These algorithms are generalized from the algorithm in [10]. We prove the convergence properties of these algorithms. Finally, we implement these algorithms and provide some examples in Section 4.

2. Conditions for the absence of an LPM duality gap

Now we formulate a linear programming problem for measure spaces (LPM). As in [10], X and Y are compact Hausdorff spaces, $C(X)$ and $M(X)$ are, respectively, spaces of continuous real valued functions and regular Borel measures on X. We denote the totality of non–negative Borel measures on X as $M^+(X)$, and the subset of $C(X)$ consisting of non–negative functions as $C^+(X)$. Given $\nu, \nu^* \in M(Y)$, $\phi \in C(X \times Y)$ and $h \in C(X)$, we consider the linear transformation $A: M(X) \to M(Y)$ defined by

$$A\mu(B) := \int_B \phi(\mu, y)d\nu^*(y) \quad \text{for any } B \in \mathcal{B}(Y) \text{ and } \mu \in M(X), \tag{1}$$

where $\mathcal{B}(Y)$ stands for the Borel field of Y, and

$$\phi(\mu, y) := \int_X \phi(x, y)d\mu(x) \in C(Y). \tag{2}$$

Also, we define the linear functional $\langle \cdot, \cdot \rangle_1$ on $M(X) \times C(X)$ as follows:

$$\langle \mu, f \rangle_1 := \int_X f(x)d\mu(x) \text{ for all } \mu \in M(X) \text{ and } f \in C(X).$$

Then we know from [10] that LPM can be formulated as follows:

LPM:minimize $\langle \mu, h \rangle_1$
subject to $\mu \in M^+(X)$,
and $A\mu \geqslant \nu$.

Moreover, we define the linear functional $\langle \cdot, \cdot \rangle_2$ on $M(Y) \times C(Y)$ as follows:

$$\langle \nu, g \rangle_2 := \int_Y g(y)d\nu(y) \text{ for all } \nu \in M(Y) \text{ and } g \in C(Y).$$

Then by applying Fubini Theorem, we have

$$\begin{aligned} \langle A\mu, g \rangle_2 &= \int_Y g(y) \int_X \phi(x, y)d\mu(x)d\nu^*(y) \\ &= \int_X \int_Y g(y)\phi(x, y)d\nu^*(y)d\mu(x) \\ &= \langle \mu, A^*g \rangle_1, \end{aligned} \tag{3}$$

where $A^*g(\cdot) := \int_Y g(y)\phi(\cdot, y)d\nu^*(y)$ is the adjoint operator of A. It is clear that $M^+(X)$ is a $\sigma(M(X), C(X))$-closed convex cone, and that $M^+(Y)$ is a $\sigma(M(Y), C(Y))$-closed convex cone. From Kretschmer (1961)[7], we know that LPM has an associated dual problem:
DLPM:

maximize $\langle \nu, g \rangle_2$
subject to $g \in C^+(Y)$,
and $A^*g \leqslant h$. (4)

From Yamasaki (1968) [17,p.344], we know that the Mackey topology $\tau(C(X), M(X))$ is the topology induced by the norm on $C(X)$ defined by

$$\|h\| := \max_{x\in X}|h(x)| \quad \text{for any } h\in C(X).$$

Now we define the set H as follows:

$$H := \{(A\mu-\overline{\nu}, \langle\mu,h\rangle_1+r): \quad \mu\in M^+(X), \overline{\nu}\in M^+(Y),\, r\geqslant 0\}.$$

A feasible solution of an optimization problem (P) is a point satisfying the constraints of problem (P). The set of all feasible solutions for problem (P) is called the feasible set of problem (P). If the feasible set of problem (P) is not empty, then we say that the problem (P) is consistent. We denote by V(P) the optimal value for problem (P). Let F(LPM) and F(DLPM) be the feasible sets for problems LPM and DLPM, respectively.

Note that if $\mu\in F$(LPM) and $g\in F$(DLPM), then $\int_X h(x)d\mu(x)\geqslant\int_Y g(y)d\nu(y)$, which entails V(LPM)$\geqslant V$(DLPM). First, we give a very simple condition under which there is no duality gap for LPM. Then we show that the existence of a Slater point for DLPM also guarantees a zero duality gap.

THEOREM 2.1. *If μ_0, the zero measure in X, is feasible for* LPM *and $h(x)\geqslant 0$ for all $x\in X$, then V*(LPM)$=V$(DLPM).

Proof. Since the function zero on Y, g_0, is feasible for DLPM, and $\int_X h(x)d\mu_0(x) =\int_Y g_0(y)d\nu(y)=0$, we have V(LPM)$=V$(DLPM).

THEOREM 2.2. *Suppose* DLPM *is consistent with finite value. If there exists a $g^*\in C^+(Y)$ such that*

$$\int_Y g^*(y)\phi(x,y)d\nu^*(y)<h(x) \quad \textit{for all}\, x\in X, \tag{5}$$

then LPM *is solvable and has no duality gap.*

Proof. Since the set $\{f\in C^+(X): f(x)>0, \forall x\in X\}$ is open in the Mackey topology $\tau(C(X),M(X))$, $C^+(X)$ will have a nonempty interior in $\tau(C(X),M(X))$. From (2.2) we know that $h(\cdot)-\int_Y g^*(y)\phi(\cdot,y)d\nu^*(y)$ is in $\tau(C(X),M(X))$–interior of $C^+(X)$, it follows that the set H is closed in $\sigma(M(Y)\times R, C(Y)\times R)$, and that LPM is solvable and has no duality gap, according to Corollary 3.1 in [7].

COROLLARY 2.3. *Suppose that* DLPM *is consistent and has a finite optimal value. If $h(x)>0$ or $h(x)<0$ for all $x\in X$, then* LPM *is solvable and has no duality gap.*

Proof. If $h(x)>0$ for all $x\in X$, then

$$\int_Y g_0(y)\phi(x,y)d\nu^*(y)<h(x) \text{for all}\, x\in X,$$

where, g_0 is the zero function on F. Thus, according to Theorem 2.2, LPM is solvable and V(LPM)= V(DLPM). On the other hand, if $h(x)<0$ for all $x\in X$, then there exists a $g^*\in C^+(Y)$ such that

$$\int_Y g^*(y)\phi(x,y)d\nu^*(y)\leqslant h(x)\,\text{for all}\,x\in X,$$

since DLPM is consistent. Let $c>1$ and define the function $g_1^*:=cg^*$, and then we have that

$$\int_Y g_1^*(y)\phi(x,y)d\nu^*(y)<h(x)\,\text{for all}\,x\in X.$$

It follows, from Theorem 2.2, that LPM is solvable and V(LPM)= V(DLPM).

EXAMPLE 2.4. *Let us consider the following elements*: $X:=\mathbb{N}\cup\{w\}$ *is the Alexandroff one point compactification of the discrete space* $\mathbb{N}$ *of all natural numbers,* $Y:=\{1,2\}$,

$$\phi(x,y):=\begin{cases}(-1)^{y+1}\dfrac{1}{x} & \textit{if}\, x\neq w,\\ 0 & \textit{if}\, x=w,\end{cases}$$

and $h(x):=1$, $\forall x\in X$.

Consider $\nu,\nu^*\in M(Y)$ such that $\nu(1):=1$, $\nu(2):=-1$, $\nu^*(y):=1$, $\forall y\in Y$. Then the LPM problem becomes

LPM:minimize $\mu(w)+\sum_{j=1}^{\infty}\mu(j)$
subject to $\mu(w),\mu(j)\geqslant 0,\forall j\in\mathbb{N}$,
and $\sum_{j=1}^{\infty}\frac{1}{j}\mu(j)=1$,

whereas its dual problem is

DLPM:maximize $g(1)-g(2)$
subject to $g(1)-g(2)\leqslant j,\forall j\in\mathbb{N}$,
and $g(1)\geqslant 0, g(2)\geqslant 0$.

It is obvious that V(DLPM) $=1$, and that the optimal solutions are those $g:\{1,2\}\to\mathbb{R}$ such that $g(2)\geqslant 0$ and $g(1)=1+g(2)$. Since $h(x)>0$ for all $x\in X$, V(LPM) $=1$ according to Corollary 2.3. Then any optimal solution of LPM,

$\mu \in M^+(X)$, must satisfy $\mu(w)+\sum_{j=1}^{\infty}\mu(j)=1$ and $\sum_{j=1}^{\infty}\frac{1}{j}\mu(j)=1$. Substracting we get $\mu+\sum_{j=2}^{\infty}(1-\frac{1}{j})\mu(j)=0$, i.e., $\mu(w)=\mu(j)=0$, $j=2,3,\ldots$. Thus the unique optimal solution of LPM is the atomic measure 1 concentrated at 1.

3. Algorithms for LPM

From now on we shall assume that the given measures ν and ν^* are absolutely continuous with respect to the Lebesgue measure m, and have density functions f^* and g^* respectively, since the most important Borel measures are those which are absolutely continuous with respect to the Lebesgue measure. Then the following Proposition 3.1 will show that the primal problem can be formulated as follows:

(P):minimize $\int_X h(x)d\mu(x)$
subject to $\mu \in M^+(X)$,
and $\phi(\mu, y)g^*(y) \geqslant f^*(y)$ a.e. in Y.

If we assume in (P) that f^* and g^* are continuous on Y, then the LPM problem can be reformulated as the general capacity problem (GCAP). Since [8] and [16] present algorithms for solving such problems, our motivation in this section is to develop algorithms for solving the LPM problem when f^* and g^* are discontinuous on Y.

PROPOSITION 3.1. *Let ν and ν^* be defined as above. Then* LPM *can be reformulated as* (P).

Proof. If μ is a feasible solution of LPM, then

$$\int_B \phi(\mu, y)g^*(y)\,dm(y) \geqslant \int_B f^*(y)dm(y) \quad \text{for all}\, B \in \mathcal{B}(Y).$$

That is,

$$\int_B [\phi(\mu, y)g^*(y) - f^*(y)]\,dm(y) \geqslant 0 \,\text{for every}\, B \in \mathcal{B}(Y).$$

Thus $\phi(\mu, y)g^*(y) \geqslant f^*(y)$ a.e. in Y w.r.t. m, and μ is also a feasible solution for problem (P). Conversely, if μ^* is feasible for problem (P), then for $B \in \mathcal{B}(Y)$, we have

$$\int_B \phi(\mu, y)g^*(y)\,dm(y) \geqslant \int_B f^*(y)dm(y).$$

This implies that

$$\int_B \phi(\mu^*, y)\, d\nu^*(y) \geqslant \nu(B).$$

Since μ^* is feasible for LPM, the result follows.

Essentially, our approach extends the method introduced in [10]. First, we develop a method for solving the LPM problem when the given density functions f^* and g^* are piecewise continuous with finitely many discontinuities. It is well-known that every function of bounded variation on a closed interval can be written as the difference of two bounded increasing functions, and every bounded increasing function can be uniformly approximated by a sequence of step functions, which are piecewise continuous with finitely many discontinuities (see Proposition 3.5). Hence we can extend our method for solving the LPM problem with density functions f^* and g^* of bounded variation. Furthermore, we develop an approach for solving the LPM with f^* simple measurable.

For simplicity in the remainder of this section, let $X := [a,b]$, and $Y := [c,d]$ with $a < b$ and $c < d$. We develop algorithms for solving the LPM problem with f^* and g^* satisfying at least one of the conditions below.

(I) g^* is continuous and f^* is piecewise continuous with finitely many discontinuities.
(II) g^* and f^* are piecewise continuous with finitely many discontinuities.
(III) g^* is continuous and f^* is of bounded variation.
(IV) f^* and g^* satisfy the following conditions:

  (a) f^*, g^* are of bounded variation,
  (b) g^* is bounded away from zero, that is, there exists an $\varepsilon > 0$ such that $\inf_{y \in Y} |g^*(y)| \geqslant \varepsilon$,
  (c) There exist compact sets $K_1 \subseteq E^+ := \{y \in [c,d] \,|\, g^*(y) > 0\}$, $K_2 \subseteq E^- := \{y \in [c,d] \,|\, g^*(y) < 0\}$ such that $m(E^+ - K_1) = 0$ and $m(E^- - K_2) =$
  0,

(V) f^* and g^* satisfy the following conditions:

  (a) g^* is continuous,
  (b) f^* is a simple measurable function.

In case (I), we assume that g^* is continuous and f^* is piecewise continuous with finite discontinuities. Lai and Wu (1994)[10] developed an algorithm for solving such a problem in which f^* is piecewise continuous with one discontinuity. For subsequent work, we now derive a general result for this case.

Now, we assume that g^* is continuous and that f^* is continuous on I_j for $j=1,2,\ldots,n$, where I_j is an interval with $m(I_j)>0$, $I_i \cap I_j = \phi$ for $i \neq j$ and $\bigcup_{i=1}^{n} I_i = Y$. Let K_i be the compact closure of I_i, then $\bigcup_{i=1}^{n} K_i = Y$. We assume that

$$f^*(y) = \begin{cases} f_1(y), & y \in I_1,\ f_1 \in C(I_1), \\ \quad \vdots & \\ f_n(y), & y \in I_n,\ f_n \in C(I_n). \end{cases} \tag{6}$$

For convenience, we continuously extend the function f_i from I_i to K_i. Then we show that the LPM problem can be reformulated as

$$\begin{aligned} (\mathrm{P}_1)\text{:minimize } & \int_X h(x)\, d\mu(x) \\ & \text{subject to } \mu \in M^+(X), \text{ and} \\ & \phi(\mu,y) g^*(y) \geqslant f_1(y),\ y \in K_1, \\ & \quad \vdots \qquad\qquad \vdots \\ & \phi(\mu,y) g^*(y) \geqslant f_n(y),\ y \in K_n. \end{aligned}$$

PROPOSITION 3.2. *Assume condition* (I). *Then the* LPM *problem can be reformulated as problem* (P_1).

Proof. We have proved that the LPM problem is equivalent to problem (P) in Proposition 3.1. Since f^* is defined in (3), we can reformulate problem (P) as follows

$$\begin{aligned} & \text{minimize } \int_X h(x) d\mu(x) \\ & \text{subject to } \mu \in M^+(X), \text{ and} \\ & \phi(\mu,y) g^*(y) \geqslant f_1(y), \quad \text{a.e. in } K_1 \\ & \quad \vdots \qquad\qquad \vdots \\ & \phi(\mu,y) g^*(y) \geqslant f_n(y), \quad \text{a.e. in } K_n. \end{aligned}$$

Since $\phi(\mu,y)g^*(y) - f_i(y)$, $i=1,\ldots,n$, are continuous functions in K_i, $i=1,\ldots,n$, respectively, $\phi(\mu,y)g^*(y) - f_i(y) \geqslant 0$, for all $y \in K_i$, $i=1,\ldots,n$. Hence we can reformulate problem LPM as problem (P_1).

Now we choose $Y^{(1)} := \{y_1^{(1)}, \cdots, y_{m_1}^{(1)}\} \subset K_1$, $Y^{(2)} := \{y_1^{(2)}, \cdots, y_{m_2}^{(2)}\} \subset K_2, \ldots,$ $Y^{(n)} := \{y_1^{(n)}, \cdots, y_{m_n}^{(n)}\} \subset K_n$, and define the semi-infinite problem SIP

$\{Y^{(1)} \cup Y^{(2)} \cup \cdots \cup Y^{(n)}\}$ as follows:

$$\begin{aligned}
&\text{minimize} \quad \int_X h(x)\,d\mu(x)\\
&\text{subject to}\ \mu \in M^+(X), \quad \text{and}\\
&\phi(\mu, y)g^*(y) \geqslant f_1(y), \quad \forall y \in Y^{(1)},\\
&\quad\ \vdots \qquad\qquad\quad \vdots\\
&\phi(\mu, y)g^*(y) \geqslant f_n(y), \quad \forall y \in Y^{(n)}.
\end{aligned}$$

Then we have the following algorithm for the present LPM.
Algorithm (1):

Step 1: Set $\ell = 1$ and choose $y_1^{(1)} \in K_1$, $y_1^{(2)} \in K_2, \ldots, y_1^{(n)} \in K_n$.

Let $Y_1^{(1)} := \{y_1^{(1)}\}$, $Y_1^{(2)} := \{y_1^{(2)}\}, \ldots, Y_1^{(n)} := \{y_1^{(n)}\}$.

Step 2: Find an optimal solution μ_ℓ for SIP $\{Y_\ell^{(1)} \cup Y_\ell^{(2)} \cup \cdots \cup Y_\ell^{(n)}\}$.

Step 3: For $i \in \{1, 2, \ldots, n\}$, find $y_{\ell+1}^{(i)} \in K_i$ such that
$y_{\ell+1}^{(i)} = \arg\min_{y \in K_i} \{\phi(\mu_\ell, y)g^*(y) - f_i(y)\}$.

Step 4: For $i \in \{1, 2, \ldots, n\}$, define
$z_{\mu_\ell}^{(i)}(y) := \phi(\mu_\ell, y)g^*(y) - f_i(y), \quad \forall y \in K_i$.

If $z_{\mu_\ell}^{(i)}(y_{\ell+1}^{(i)}) \geqslant 0$ for every $i \in \{1, 2, \ldots, n\}$, then stop, and μ_ℓ is an optimal solution for LPM. Otherwise continue.

Step 5: Set $Y_{\ell+1}^{(i)} := Y_\ell^{(i)} \cup \{y_{\ell+1}^{(i)}\}$, $i = 1, 2, \ldots, n$.

(But if $y_{\ell+1}^{(i)} \in Y_\ell^{(i)}$ then $y_{\ell+1}^{(i)}$ would not enter $Y_\ell^{(i)}$ repeatedly.)
Update ℓ by $\ell + 1$ and go to Step 2.

THEOREM 3.3. *Let $\{\mu_\ell\}_{\ell=1}^{\infty}$ be a sequence of optimal solutions for the* SIP *problem generated by using the above procedure. If some $M > 0$ exists such that $\|\mu_\ell\| \leqslant M$ for all ℓ, then there will be a subsequence $\{\mu_{\ell_i}\}_{i=1}^{\infty}$ of $\{\mu_\ell\}_{\ell=1}^{\infty}$ that converges to an optimal solution μ^* of the* LPM.

Proof. Using an argument similar to that used in [10]Theorem 4.3, we can obtain this result.

In case (II), we assume that g^* and f^* are piecewise continuous with finite discontinuities. Without loss of generality, we may assume that there exist intervals

$I_1, I_2, \ldots, I_n$ such that

$$g^*(y) = \begin{cases} g_1(y), & y \in I_1 \\ \vdots & \\ g_n(y), & y \in I_n \end{cases}, \quad \text{where } g_i(y) \in C(I_i) \quad i = 1, 2, \ldots, n,$$

and

$$f^*(y) = \begin{cases} f_1(y), & y \in I_1 \\ \vdots & \\ f_n(y), & y \in I_n \end{cases}, \quad \text{where } f_i(y) \in C(I_i) \quad i = 1, 2, \ldots, n.$$

For convenience, we continuously extend the functions f_i and g_i from I_i to its closure K_i. Then the LPM problem can be reformulated as

$$\begin{aligned} &\text{minimize } \int_X h(x)\, d\mu(x) \\ &\text{subject to } \mu \in M^+(X), \text{ and} \\ &\quad \phi(\mu, y) g_1(y) \geqslant f_1(y), \quad y \in K_1 \\ &\quad \vdots \qquad\qquad \vdots \\ &\quad \phi(\mu, y) g_n(y) \geqslant f_n(y), \quad y \in K_n. \end{aligned}$$

In order to solve the LPM of case (II), we can use a method similar to Algorithm (1).

In case (III), we assume that g^* is continuous on Y, and f^* has bounded variation on Y. For any given function S on Y, we define the problem LPM–S by replacing f^* by S in (P). Thus, the LPM–S has the following form.

$$\begin{aligned} &\text{LPM–}S\text{:minimize } \int_X h(x)\, d\mu(x) \\ &\text{subject to } \mu \in M^+(X), \text{ and} \\ &\qquad \phi(\mu, y) g^*(y) \geqslant S(y), \text{ a.e. in } Y. \end{aligned}$$

The following lemma provides a key result for subsequent work.

LEMMA 3.4. *If there exists a sequence of functions* $\{S_n\}_{n=1}^{\infty}$ *and* $\overline{\mu} \in M^+(X)$ *such that*

(i) $S_1(y) \leqslant S_2(y) \leqslant \cdots \leqslant f^*(y)$ *a.e.,*

(ii) $S_n(y) \to f^*(y)$ *uniformly a.e., as* $n \to \infty$, *and*

(iii) $\inf_{y \in Y} \phi(\overline{\mu}, y) g^*(y) > 0$,

then $V(\text{LPM}) = \lim_{n\to\infty} V(\text{LPM–}S_n)$.

Proof. Let $\delta := \inf_{y \in Y} \phi(\overline{\mu}, y) g^*(y)$. Then $\delta > 0$, according to the assumption *(iii)*. Thus, $\frac{1}{\delta}\phi(\overline{\mu}, y) g^*(y) \geqslant 1$ for any $y \in Y$. Since $S_n(y) \to f^*(y)$ uniformly a.e. in Y, as $n \to \infty$, it follows that for any $\varepsilon > 0$ there exists an $N_\varepsilon \in \mathbb{N}$ such that $|S_{N_\varepsilon}(y) - f^*(y)| \leqslant \varepsilon$ a.e. in Y. Let $\mu_{N_\varepsilon} \in F(\text{LPM–}S_{N_\varepsilon})$ be such that

$$\int_X h\, d\mu_{N_\varepsilon} < V(\text{LPM–}S_{N_\varepsilon}) + \varepsilon. \tag{7}$$

Then

$$\begin{aligned} \phi(\mu_{N_\varepsilon} + \frac{\varepsilon}{\delta}\overline{\mu}, y) g^*(y) &= \phi(\mu_{N_\varepsilon}, y) g^*(y) + \frac{\varepsilon}{\delta}\phi(\overline{\mu}, y) g^*(y) \\ &\geqslant S_{N_\varepsilon}(y) + \varepsilon \\ &\geqslant f^*(y) \quad \text{a.e. in } Y, \end{aligned} \tag{8}$$

and this implies that $\mu_{N_\varepsilon} + \frac{\varepsilon}{\delta}\overline{\mu} \in F(\text{LPM})$. Hence

$$V(\text{LPM}) \leqslant \int_X h\, d(\mu_{N_\varepsilon} + \frac{\varepsilon}{\delta}\overline{\mu}).$$

Since $V(\text{LPM–}S_{N_\varepsilon}) \leqslant V(\text{LPM})$, and according to (7), we have

$$\begin{aligned} 0 &\leqslant V(\text{LPM}) - V(\text{LPM–}S_{N_\varepsilon}) \\ &\leqslant \int_X h\, d(\mu_{N_\varepsilon} + \frac{\varepsilon}{\delta}\overline{\mu}) - \int_X h\, d\mu_{N_\varepsilon} \\ &\leqslant \frac{\varepsilon}{\delta} \Big|\int_X h\, d\overline{\mu}\Big| + \varepsilon. \end{aligned} \tag{9}$$

Taking $\varepsilon \to 0$ in (9), it follows that $V(\text{LPM}) = \lim_{n\to\infty} V(\text{LPM–}S_n)$.

Given an arbitrary function f^* on Y, we may define a truncation function $T_n(f^*)$ for any $n \in \mathbb{N}$ as follows:

$$T_n(f^*)(y) := \begin{cases} n, & \text{if } f^*(y) \geqslant n, \\ \dfrac{k-1}{10^n}, & \text{if } \dfrac{k-1}{10^n} \leqslant f^*(y) < \dfrac{k}{10^n}, \\ & \quad k = 0, 1, 2, \ldots, n \cdot 10^n \\ & \qquad -1, -2, \ldots, -n \cdot 10^n + 1 \\ -n, & \text{if } f^*(y) < -n. \end{cases} \tag{10}$$

Then it will be easy to establish the following result:

PROPOSITION 3.5.

(*i*) *If* f^* *is bounded from below with a lower bound* m_{f^*}, *then* $T_n(f^*)(y) \leqslant T_{n+1}(f^*)(y) \leqslant f^*(y)$ *for all* $n \geqslant -m_{f^*}$, *and* $T_n(f^*)(y) \to f^*(y)$ *for all* $y \in Y$ *as* $n \to \infty$.

(*ii*) *If for some* $M > 0$, $|f^*(y)| \leqslant M$ *for all* $y \in Y$, *then* $\|f^* - T_n(f^*)\|_\infty := \sup\limits_{y \in Y} |f^*(y) - T_n(f^*)(y)| \leqslant \frac{1}{10^n}$ *for all* $n \geqslant M$. *Thus,* $T_n(f^*) \to f^*$ *uniformly on* Y *as* $n \to \infty$.

(*iii*) *If* f^* *is monotone, then* $T_n(f^*)$ *is a step function, for every* n.

Now, if f^* is a bounded variation function on Y, then f^* can be written as a difference of two monotone increasing functions g_1 and g_2. To derive an algorithm for solving the LPM of case (III), we define T_n^* on bounded variation function f^* by

$$T_n^*(f^*)(y) := T_n(g_1)(y) + T_n(-g_2)(y), \tag{11}$$

where $f^* = g_1 - g_2$ and g_1, g_2 are monotone increasing functions on Y. Then according to Lemma 3.4, we have the following theorem:

THEOREM 3.6. *For the* LPM *problem, we assume that* g^* *is continuous on* Y *and* f^* *is of bounded variation on* Y. *If there exists a* $\overline{\mu} \in M^+(X)$ *such that*

$$\inf_{y \in Y} \phi(\overline{\mu}, y) g^*(y) > 0,$$

then $V(\text{LPM}) = \lim\limits_{n \to \infty} V(\text{LPM–}T_n^*(f^*))$.

Proof. Since f^* is of bounded variation, f^* is also bounded. So, according to Proposition 3.5, there exists a $N \in \mathbb{N}$ such that if $n \geqslant N$, then

$$T_n^*(f^*)(y) \leqslant T_{n+1}^*(f^*)(y) \leqslant f^*(y), \quad \text{and}$$
$$\|T_n(g_1) - g_1\|_\infty \leqslant \frac{1}{10^n}, \quad \|T_n(-g_2) - (-g_2)\|_\infty \leqslant \frac{1}{10^n},$$

where g_1, g_2 are monotone increasing functions such that $f^* = g_1 - g_2$. Hence,

$$\begin{aligned} \|T_n^*(f^*) - f^*\|_\infty &\leqslant \|T_n(g_1) - g_1\|_\infty + \|T_n(-g_2) - (-g_2)\ |_\infty \\ &\leqslant \frac{2}{10^n}, \quad \text{for all } n \geqslant \mathbb{N}. \end{aligned}$$

That is,

$$T_n^*(f^*) \to f^* \quad \text{uniformly on } Y \text{ as } n \to \infty.$$

Therefore, according to Lemma 3.4, $V(\text{LPM}) = \lim_{n\to\infty} V(\text{ LPM–}T_n^*(f^*))$.

Note that, according to Proposition 3.5–(iii), $T_n^*(f^*)$ is a step function, and therefore every LPM–$T_n^*(f^*)$ is of the type described in case (I). By this property, we derive an iterative process for solving the LPM problem of case (III). First, we find a $k \in \mathbb{N}$ large enough such that $T_k^*(f^*)(y) \leqslant f^*(y)$. Next we consider the problem LPM–$T_k^*(f^*)$ and assume that LPM–$T_k^*(f^*)$ is solvable. Since it is of the type described in case (I), we can solve it by using Algorithm (1). Suppose that μ_k^* is an optimal solution of LPM–$T_k^*(f^*)$. Calculate

$$r_k := \inf_{y\in Y}\{\phi(\mu_k^*, y)g^*(y) - f^*(y)\}. \tag{12}$$

(a) If $r_k \geqslant 0$, then μ_k^* will be an optimal solution to the LPM. Since μ_k^* is feasible for the LPM,

$$V(\text{LPM}) \leqslant \int_X h(x)\, d\mu_k^*(x) = V(\text{LPM–}T_k^*(f^*)).$$

Thus,

$$V(\text{LPM}) = V(\text{LPM–}T_k^*(f^*))$$

because it is obvious that $V(\text{LPM–}T_k^*(f^*)) \leqslant V(\text{LPM})$.

(b) If $r_k < 0$, then there exists an $N_k \geqslant k$ such that

$$\inf_{y\in Y}\{\phi(\mu_k^*, y)g^*(y) - T_{N_k}^*(f^*)(y)\} \geqslant 0,$$

but

$$\inf_{y\in Y}\{\phi(\mu_k^*, y)g^*(y) - T_{N_k+1}^*(f^*)(y)\} < 0$$

because, otherwise, for each $y \in Y$

$$\phi(\mu_k^*, y)g^*(y) \geqslant T_n^*(f^*)(y), \quad \forall n \geqslant k.$$

Let $n \to \infty$, and then we have

$$\phi(\mu_k^*, y)g^*(y) \geqslant f^*(y), \quad \forall y \in Y.$$

From (12), we have $r_k \geqslant 0$ and this contradicts the assumption that $r_k < 0$. Thus case (b) holds. From case (b), we know that μ_k^* is optimal for LPM–$T_{N_k}^*(f^*)$ but not feasible for LPM–$T_{N_k+1}^*(f^*)$. If we update k by using $N_k + 1$ and continue the process, we get an iterative procedure for solving the present LPM.

We now summarize the process described above in the following:

Algorithm (2):

Step 1: We find $k \in \mathbb{N}$ and $k \geqslant \max\{\|g_1\|_\infty, \|g_2\|_\infty\}$, where g_1, g_2 are monotone increasing functions such that $f^* = g_1 - g_2$.

Step 2: Solve the LPM–$T_k^*(f^*)$, where $T_k^*(f^*)$ is defined in (3), and obtain an optimal solution μ_k^*.

Step 3: Compute

$$r_k := \inf_{y \in Y}\{\phi(\mu_k^*, y)g^*(y) - f^*(y)\}.$$

If $r_k \geqslant 0$, then stop !! And μ_k will be an optimal solution for the LPM. Otherwise continue.

Step 4: Find $N_k \geqslant k$ such that

$$\inf_{y \in Y}\{\phi(\mu_k^*, y)g^*(y) - T_{N_k}^*(f^*)(y)\} \geqslant 0$$

but

$$\inf_{y \in Y}\{\phi(\mu_k^*, y)g^*(y) - T_{N_k+1}^*(f^*)(y)\} < 0.$$

Step 5: Update k by using $N_k + 1$ and go to step 2.

In the above algorithm, we assume that every LPM–$T_k^*(f^*)$ is solvable.

Theorem 3.6 shows that V(LPM) can be approximated by using V (LPM–$T_n^*(f^*)$). The following theorem provides an error bound for this approximation.

THEOREM 3.7. *Suppose there exists a* $\overline{\mu} \in M^+(X)$ *such that*

$$\rho := \inf_{y \in Y} \phi(\overline{\mu}, y)g^*(y) > 0.$$

(i) *If* $\int_X h(x)\,d\overline{\mu}(x) < 0$, *then* $V(\text{LPM}) = -\infty$.

(ii) *If* $\int_X h(x)\,d\overline{\mu}(x) \geqslant 0$, *then*

$$0 \leqslant V(\text{LPM}) - V(\text{LPM–}T_n^*(f^*)) \leqslant \frac{1}{\rho} \cdot \frac{2}{10^n} \int_X h(x)\,d\overline{\mu}(x)$$

for all $n \geqslant \max\{\|g_1\|_\infty, \|g_2\|_\infty\}$, *where* g_1 *and* g_2 *are monotone increasing functions such that* $f^* = g_1 - g_2$.

Proof.

(i) Since $\rho := \inf_{y\in Y} \phi(\overline{\mu}, y) g^*(y) > 0$. we can take $k \in \mathbb{N}$ such that $k\rho > \sup_{y\in Y} f^*(y)$, it follows that $k\overline{\mu} \in F(\text{LPM})$. Hence

$$V(\text{LPM}) \leqslant k \int_X h(x)\, d\overline{\mu}(x).$$

Let $k \to \infty$, and we have that $V(\text{LPM}) = -\infty$.

(ii) If $n \geqslant \max\{\|g_1\|_\infty, \|g_2\|_\infty\}$, then, by repeating the argument of Theorem 3.6, we have

$$T_n^*(f^*)(y) := T_n(g_1)(y) + T_n(-g_2)(y) \leqslant g_1(y) - g_2(y) = f^*(y), \quad \text{for each } y \text{ in } Y, \tag{13}$$

and

$$\begin{aligned}\|T_n^*(f^*) - f^*\|_\infty &\leqslant \|T_n(g_1) - g_1\|_\infty + \|T_n(-g_2) - (-g_2)\|_\infty \\ &\leqslant \frac{2}{10^n}.\end{aligned}$$

Hence $T_n^*(f^*)(y) + \frac{2}{10^n} \geqslant f^*(y)$ for each y in Y. Then, for each $\mu \in F(\text{LPM–}T_n^*(f^*))$, we have

$$\begin{aligned}\phi\Big(\mu + \frac{2}{10^n\rho}\overline{\mu}, y\Big) g^*(y) &\geqslant T_n^*(f^*)(y) + \frac{2}{10^n} \quad \text{a.e. in } Y \\ &\geqslant f^*(y) \quad \text{a.e. in } Y.\end{aligned}$$

That is, $\mu + \frac{2}{10^n\rho}\overline{\mu} \in F(\text{LPM})$. Therefore, for every $\mu \in F(\text{LPM–}T_n^*(f^*))$,

$$V(\text{LPM}) \leqslant \int_X h(x)\, d\mu(x) + \frac{2}{10^n\rho}\int_X h(x)\, d\overline{\mu}(x).$$

This implies

$$V(\text{LPM}) - V(\text{LPM–}T_n^*(f^*)) \leqslant \frac{2}{10^n\rho}\int_X h(x)\, d\overline{\mu}(x),$$

for $n \geqslant \max\{\|g_1\|_\infty, \|g_2\|_\infty\}$. Since, according to (3), $T_n^*(f^*) \leqslant f^*$, it follows that

$$V(\text{LPM–}T_n^*(f^*)) \leqslant V(\text{LPM}).$$

Thus,

$$0 \leqslant V(\text{LPM}) - V(\text{LPM–}T_n^*(f^*)) \leqslant \frac{2}{10^n\rho}\int_X h(x)\, d\overline{\mu}(x).$$

Next, we consider LPM of case (IV). It is obvious that LPM can be rewritten in the following form:

$$\begin{aligned} &\text{minimize} \quad \int_X h(x)\,d\mu(x) \\ &\text{subject to}\, \mu \in M^+(X), \quad \text{and} \\ &\phi(\mu, y) \geqslant \frac{f^*}{g^*}(y) \quad \text{a.e. in } K_1, \\ &-\phi(\mu, y) \geqslant -\frac{f^*}{g^*}(y) \quad \text{a.e . in } K_2. \end{aligned}$$

Since f^* and g^* are of bounded variation and g^* is bounded away from zero, it follows that $\frac{f^*}{g^*}$ is of bounded variation.

Let $\frac{f^*}{g^*} := h_1 - h_2$, where h_1, h_2 are monotone increasing on $[c, d]$. As in (10) and (11), we define

$$T_n^*(\frac{f^*}{g^*}) := T_n(h_1) + T_n(-h_2),$$

and

$$T_n^*(-\frac{f^*}{g^*}) := T_n(h_2) + T_n(-h_1), \quad \forall n \in \mathbb{N}.$$

To solve LPM in case (IV), we define the subprogram LPM–$T_n^*(\frac{f^*}{g^*}) \cup T_n^*(-\frac{f^*}{g^*})$ for $n \in \mathbb{N}$ as follows:

$$\begin{aligned} &\text{minimize} \quad \int_X h(x)\,d\mu(x) \\ &\text{subject to}\, \mu \in M^+(X), \quad \text{and} \\ &\phi(\mu, y) \geqslant T_n^*(\frac{f^*}{g^*})(y) \quad \text{a.e. in } K_1, \\ &-\phi(\mu, y) \geqslant T_n^*(-\frac{f^*}{g^*})(y) \quad \text{a.e. in } K_2. \end{aligned}$$

The following theorem shows that V(LPM) can be approximated by using V (LPM–$T_n^*(\frac{f^*}{g^*}) \cup T_n^*(-\frac{f^*}{g^*})$). Note that we can solve every subprogram LPM–$T_n^*(\frac{f^*}{g^*}) \cup T_n^*(-\frac{f^*}{g^*})$ by using Algorithm (1).

THEOREM 3.8. *Suppose that there exists $\mu^* \in M^+(X)$ such that $\phi(\mu^*, y)g^*(y) \geqslant 1$ a.e. in Y.*

(*i*) *If $\int_X h(x)\,d\mu^*(x) < 0$, then $V(\mathrm{LPM}) = -\infty$.*
(*ii*) *If $\int_X h(x)\,d\mu^*(x) \geqslant 0$, then*

$$\begin{aligned} &V(\mathrm{LPM}\text{–}T_n^*(\frac{f^*}{g^*}) \cup T_n^*(-\frac{f^*}{g^*})) \\ &\leqslant V(\mathrm{LPM}) \leqslant V(\mathrm{LPM}\text{–}T_n^*(\frac{f^*}{g^*}) \cup T_n^*(-\frac{f^*}{g^*})) + \frac{2k}{10^n}\int_X h(x)\,d\mu^*(x) \end{aligned}$$

for all $n \geqslant \max\{\|h_1\|_\infty, \|h_2\|_\infty\}$, and $k := \sup_{y\in Y}|g^*(y)|$.

Proof.

(i) It follows, by using an argument similar to that we used to prove Theorem 3.7–(i).
(ii) Since $\phi(\mu^*, y)g^*(y) \geqslant 1$ a.e. in Y, and $0 < |g^*(y)| \leqslant k\ \forall y \in Y$, it follows that

$$\phi(\mu^*, y) \geqslant \frac{1}{g^*(y)} \geqslant \frac{1}{k} \quad \text{a.e. in } K_1, \tag{14}$$

and

$$-\phi(\mu^*, y) \geqslant \frac{-1}{g^*(y)} \geqslant \frac{1}{k} \quad \text{a.e. in } K_2. \tag{15}$$

Thus, if $n \geqslant \max\{\|h_1\|_\infty, \|h_2\|_\infty\}$, and according to Proposition 3.5, we have that

$$\begin{aligned} T_n^*(\frac{f^*}{g^*})(y) &:= T_n(h_1)(y) + T_n(-h_2)(y) \\ &\leqslant (h_1 - h_2)(y) = \frac{f^*}{g^*}(y), \end{aligned} \tag{16}$$

$$\begin{aligned} T_n^*(-\frac{f^*}{g^*})(y) &:= T_n(h_2)(y) + T_n(-h_1)(y) \\ &\leqslant (h_2 - h_1)(y) = -\frac{f^*}{g^*}(y), \end{aligned} \tag{17}$$

and

$$\begin{aligned} \|T_n^*(\frac{f^*}{g^*}) - \frac{f^*}{g^*}\|_\infty &\leqslant \|T_n(h_1) - h_1\|_\infty + \|T_n(-h_2) - (-h_2)\|_\infty \\ &\leqslant \frac{1}{10^n} + \frac{1}{10^n} = \frac{2}{10^n}. \end{aligned} \tag{18}$$

Likewise,

$$\left\| T_n^*(-\frac{f^*}{g^*}) - (-\frac{f^*}{g^*}) \right\|_\infty \leqslant \frac{2}{10^n}. \tag{19}$$

Therefore, if $\mu \in F(\text{LPM–}T_n^*(\frac{f^*}{g^*}) \cup T_n^*(-\frac{f^*}{g^*}))$, and for $y \in K_1$,

$$\begin{aligned}
&\phi(\mu + \frac{2k}{10^n}\mu^*, y) \\
&\quad \geqslant T_n^*(\frac{f^*}{g^*})(y) + \frac{2k}{10^n}\phi(\mu^*, y) \quad \text{a .e. in } K_1 \\
&\quad \geqslant T_n^*(\frac{f^*}{g^*})(y) + \frac{2}{10^n} \quad \text{a.e. in } K_1 \text{ (by (14))} \\
&\quad \geqslant T_n^*(\frac{f^*}{g^*})(y) + \frac{f^*}{g^*}(y) - T_n^*(\frac{f^*}{g^*})(y) \quad \text{a.e. in } K_1 \text{ (from (19))} \\
&\quad = \frac{f^*}{g^*}(y), \quad \text{a.e. in } K_1
\end{aligned}$$

and, for $y \in K_2$,

$$\begin{aligned}
-\phi(\mu + \frac{2k}{10^n}\mu^*, y) &= -\phi(\mu, y) + \frac{2k}{10^n}(-\phi(\mu^*, y)) \quad \text{a.e. in } K_2 \\
&\geqslant T_n^*(-\frac{f^*}{g^*})(y) + \frac{2}{10^n} \quad \text{a.e. in } K_2 \text{ (by (15))} \\
&\geqslant -\frac{f^*}{g^*}(y). \quad \text{a.e. in } K_2 \text{ (from (19))},
\end{aligned}$$

which implies that $\mu + \frac{2k}{10^n}\mu^* \in F(\text{LPM})$. Thus,

$$V(\text{LPM}) \leqslant V(\text{LPM–}T_n^*(\frac{f^*}{g^*}) \cup T_n^*(-\frac{f^*}{g^*})) + \frac{2k}{10^n}\int_X h(x)\,d\mu^*(x).$$

From (16) and (17), we have that

$$F(\text{LPM}) \subseteq F(\text{LPM–}T_n^*(\frac{f^*}{g^*}) \cup T_n^*(-\frac{f^*}{g^*})), \forall n \geqslant \max\{\|h_1\|_\infty, \|h_2\|_\infty\},$$

which leads to

$$V(\text{LPM–}T_n^*(\frac{f^*}{g^*}) \cup T_n^*(-\frac{f^*}{g^*})) \leqslant V(\text{LPM}).$$

Consider LPM in case (V), in which f^* and g^* satisfy the following conditions:

(i) g^* is continuous on Y;

(ii) $f^* := \sum_{i=1}^{\ell} a_i \chi_{E_i}$, where E_i is a measurable set; *i.e.*, f^* is a simple function on Y.

For $n \in \mathbb{N}$, by the regularity of Lebesgue measure (see [13]), we consider the compact sets $K_i^{(n)} \subseteq E_i$, $\forall i \in \{1,2,\ldots,\ell\}$, with $\sum_{i=1}^{\ell} m(E_i - K_i^{(n)}) \leqslant \frac{1}{n}$ and $K_i^{(n)} \subseteq K_i^{(n+1)}$. As in the previous cases, we define a subprogram LPM–$V_n(f^*)$ as follows:

$$\begin{aligned}
&\text{minimize} \quad \int_X h(x)\, d\mu(x)\\
&\text{subject to}\, \mu \in M^+(X),\\
&\qquad\qquad \phi(\mu, y) g^*(y) \geqslant a_i \quad \text{a.e. in } K_i^{(n)},\ i = 1,2,\ldots,\ell.
\end{aligned}$$

Every LPM–$V_n(f^*)$ is of the type described in case (I). Let us introduce a condition for $\phi(x,y)$:

Condition $(*)$: If $\{\mu_n\}_{n=1}^{\infty}$, $\mu \in M^+(X)$ and $\mu_n \to \mu$ weakly, then

$$\max_{y\in Y} |\phi(\mu_n - \mu, y)| \to 0, \text{ as } n \to \infty.$$

Note that if ϕ is separable, that is, $\phi(x,y) := \sum_{i=1}^{k} f_i(x) g_i(y)$, where $f_i(x) \in C(X)$ and $g_i(y) \in C(Y)$, then ϕ satisfies the condition $(*)$. The following theorem shows that under certain conditions, V(LPM) can be approximated by V(LPM–$V_n(f^*)$).

THEOREM 3.9. *Let ν_n be an optimal solution for* LPM– $V_n(f^*)$, $n \in \mathbb{N}$. *If there exist $N \in \mathbb{N}$ and $k \in \mathbb{R}$ such that $\|\nu_n\| \leqslant k$ for all $n \geqslant N$, and $\phi(x,y)$ satisfies the condition* $(*)$, *then* $\lim_{n\to\infty} V(\text{LPM–}V_n(f^*)) = V(\text{LPM})$.

Proof. Since $\forall i \in \{1,2,\ldots,\ell\}$ and $n \in \mathbb{N}$, $K_i^{(n)} \subseteq K_i^{(n+1)} \subseteq E_i$, F(LPM–$V_1(f^*)) \supseteq F$(LPM–$V_2(f^*)) \supseteq \cdots \supseteq F$(LPM). So

$$V(\text{LPM–}V_1(f^*)) \leqslant V(\text{LPM–}V_2(f^*)) \leqslant \cdots \leqslant V(\text{LPM}).$$

Let $\lim_{n\to\infty} V(\text{LPM–}V_n(f^*)) = \alpha$. Hence $\alpha \leqslant V$(LPM). Since $C(X)$ is separable, and by the Banach-Alaoglu Theorem and Theorem 3.16 in [14], the set $\{\nu \in M(X) : \|\nu\| \leqslant k\}$ is sequentially compact. Hence, by the assumption that $\|\nu_n\| \leqslant k$, for all $n \geqslant N$, there exists a subsequence $\{\nu_{n_j}\}_{j=1}^{\infty} \subseteq \{\nu_n\}_{n=1}^{\infty}$ and $\nu^* \in M^+(X)$

such that $\nu_{n_j} \to \nu^*$ weakly. Therefore,

$$|\int_X h d\nu^* - \alpha| \leqslant |\int_X h d\nu^* - \int_X h d\nu_{n_j}| + |\int_X h d\nu_{n_j} - \alpha| \to 0, \quad \text{as } j \to \infty.$$

This implies that

$$\int_X h d\nu^* = \alpha \leqslant V(\text{LPM}). \tag{20}$$

We claim that $\nu^* \in F(\text{LPM–}V_n(f^*))$ for every $n \in \mathbb{N}$. Otherwise, there exists a $n^* \in \mathbb{N}$ such that $\nu^* \notin F(\text{LPM–}V_{n^*}(f^*))$, entailing the existence of a set $S \subseteq K_{i^*}^{(n^*)}$ for some $i^* \in \{1, 2, \ldots, \ell\}$ with $m(S) \neq 0$ such that

$$\phi(\nu^*, y) g^*(y) < a_{i^*}, \quad \forall y \in S.$$

Since $\phi(x, y)$ satisfies condition $(*)$ and $\nu_{n_j} \longrightarrow \nu^*$ weakly, together with the continuity of g^*,

$$\max_{y \in Y} |\phi(\nu_{n_j} - \nu^*, y) g^*(y)| \to 0, \quad \text{as } j \to \infty.$$

Hence, there exists a sufficiently large N with $N > n^*$ such that

$$\phi(\nu_N, y) g^*(y) < a_{i^*}, \quad \forall y \in S. \tag{21}$$

Since $N > n^*$, $\nu_N \in F(\text{LPM–}V_{n^*}(f^*))$, which implies that

$$\phi(\nu_N, y) g^*(y) \geqslant a_{i^*} \quad \text{a.e. in } K_{i^*}^{(n^*)}.$$

Therefore,

$$\phi(\nu_N, y) g^*(y) \geqslant a_{i^*} \quad y \in S \text{ a.e.}$$

But this contradicts (3), and so, the claim must be true. Thus, $\nu^* \in F(\text{LPM–}V_n(f^*))$ for every $n \in \mathbb{N}$. Now, using this claim we will show that $\nu^* \in F(\text{LPM})$. Suppose that this is not the case. Then there must exist a set $S' \subseteq E_{\bar{i}}$ for some $\bar{i} \in \{1, 2, \ldots, \ell\}$ with $m(S') \neq 0$ such that

$$\phi(\nu^*, y) g^*(y) < a_{\bar{i}} \quad \forall y \in S'. \tag{22}$$

Since $m(E_{\bar{i}} - \bigcup_{n=1}^{\infty} K_{\bar{i}}^{(n)}) = 0$ and $m(S') \neq 0$, there exists $K_{\bar{i}}^{(n_0)}$ for some $n_0 \in \mathbb{N}$ such that $m(K_{\bar{i}}^{(n_0)} \cap S') \neq 0$. According to the claim, $\nu^* \in F(\text{LPM–}V_{n_0}(f^*))$. Hence $\phi(\nu^*, y) g^*(y) \geqslant a_{\bar{i}}$ for $y \in K_{\bar{i}}^{(n_0)}$ a.e., which contradicts (3). Therefore, $\nu^* \in F(\text{LPM})$ and $V(\text{LPM}) = \alpha = \lim_{n \to \infty} V(\text{LPM–}V_n(f^*))$ from (3).

4. Numerical examples

In this section, we use three examples to illustrate the proposed algorithms and solution procedures. Throughout these examples, we let $X = Y = [0,1]$, and let m and δ_x be the Lebesgue measure and the unit mass concentrated at x, respectively.

EXAMPLE 4.1. *Consider the LPM with* $h(x) := 1.5 - x$, $\phi(x,y) := \cos(2\pi xy)$, *and* ν^*, ν *absolutely continuous with respect to m, having the density functions*

$$g^*(y) := 1, \quad \forall y \in Y,$$
$$f^*(y) := \sum_{n=1}^{\infty} \frac{1}{10^{n-1}} \chi_{[\frac{1}{2n}, \frac{1}{2n-1}]}(y), \quad \forall y \in Y, \quad \text{respectively.}$$

It is clear that g^* is continuous on Y. As f^* is monotone increasing, it is of bounded variation on Y, and we write $f^* = g_1 - g_2$, where $g_1 = f^*$ and $g_2 = 0$. So, by (3),

$$T_n^*(f^*)(y) = \sum_{\ell=1}^{n+1} \frac{1}{10^{\ell-1}} \chi_{[\frac{1}{2\ell}, \frac{1}{2\ell-1}]}(y), \quad y \in Y.$$

As δ_0, the unit mass concentrated at 0, is such that

$$\inf_{y\in Y}\{\phi(\delta_0, y) g^*(y)\} = \inf_{y\in Y}\{\int_X \phi(x,y)\, d\delta_0(x) g^*(y)\} = 1 > 0,$$

the optimal value of LPM can be approximated by the optimal value of LPM–$T_n^*(f^*)$, which is defined in Section 3, by applying Theorem 3.6.

Reformulate LPM as follows:

$$\begin{aligned}
&\text{minimize} \quad \int_X (1.5 - x)\, d\mu(x)\\
&\text{subject to } \mu \in M^+(X), \quad \text{and}\\
&\text{and} \int_X \cos(2\pi xy)\, d\mu(x) \geqslant \sum_{n=1}^{\infty} \frac{1}{10^{n-1}} \chi_{[\frac{1}{2n}, \frac{1}{2n-1}]}, \quad \text{a.e. in } Y.
\end{aligned}$$

As it is of the type (III) of LPM as it was introduced in Section 3, we shall solve it by Algorithm (2). Since $\max\{\|g_1\|_\infty, \|g_2\|_\infty\} = 1$, we first consider the subproblem LPM–$T_1^*(f^*)$ which is of type (II), also introduced in Section 3, and is defined as follows:

$$\begin{aligned}
\text{LPM–}T_1^*(f^*):\quad & \text{minimize} \quad \int_X (1.5-x)\,d\mu(x) \\
& \text{subject to } \mu \in M^+(X), \quad \text{and} \\
& \text{and} \int_X \cos(2\pi xy)\,d\mu(x) \geqslant \sum_{\ell=1}^{2} \frac{1}{10^{\ell-1}} \chi_{[\frac{1}{2\ell},\frac{1}{2\ell-1}]}(y), \forall y \in Y.
\end{aligned}$$

Now, we solve LPM–$T_1^*(f^*)$ by Algorithm (1). Let

$$K_1 := [\frac{1}{2}, 1], \quad K_2 := [\frac{1}{3}, \frac{1}{2}], \quad K_3 := [\frac{1}{4}, \frac{1}{3}], \quad K_4 := [0, \frac{1}{4}],$$

and define

$$\begin{aligned}
& f_1^*(y) := 1, && \forall y \in K_1, \\
& f_2^*(y) := 0, && \forall y \in K_2, \\
& f_3^*(y) := 0.1, && \forall y \in K_3, \\
& f_4^*(y) := 0, && \forall y \in K_4.
\end{aligned}$$

Then LPM–$T_1^*(f^*)$ can be rewritten as follows:

$$\begin{aligned}
& \text{minimize} \quad \int_X (1.5-x)\,d\mu(x) \\
& \text{subject to } \mu \in M^+(X), \quad \text{and} \\
& \text{and} \int_X \cos(2\pi xy)\,d\mu(x) \geqslant f_i^*(y) \quad \forall y \in K_i, \quad i = 1, 2, 3, 4.
\end{aligned}$$

As in Algorithm (1), we let, for $i = 1, 2, 3, 4$,

$$Z_{\mu_\ell^*}^{(i)}(y) := \phi(\mu_\ell^*, y) g^*(y) - f_i^*(y), \quad \forall y \in K_i,$$

and,

$$y_{\ell+1}^{(i)} = \arg\min_{y \in K_i} Z_{\mu_\ell^*}^{(i)}(y).$$

The progress of the Algorithm (1) for solving LPM–$T_1^*(f^*)$ is summarized in Table 1.

Table 1.

ℓ	Y_l	[1]$O.S.(\ell)$	$\min_{y\in K_1} Z^{(1)}_{\mu^*_\ell}$ / $y^{(1)}_{\ell+1}$	$\min_{y\in K_2} Z^{(2)}_{\mu^*_\ell}$ / $y^{(2)}_{\ell+1}$	$\min_{y\in K_3} Z^{(3)}_{\mu^*_\ell}$ / $y^{(3)}_{\ell+1}$	$\min_{y\in K_4} Z^{(4)}_{\mu^*_\ell}$ / $y^{(4)}_{\ell+1}$
1	$\{y^{(1)}_0, y^{(1)}_1\}$	[3]μ^*_1	−0.00041	1	0.903338	1.005751
			0.582382	−−	−−	−−
2	$\{y^{(1)}_0, y^{(1)}_1, y^{(1)}_2\}$	[4]μ^*_2	−0.000002	1.000423	0.903530	1.005748
			0.589055	−−	−−	−−
3	[2]$\{y^{(1)}_0, y^{(1)}_1, y^{(1)}_2, y^{(1)}_3\}$	[5]μ^*_3	0	1.000426	0.903542	1.005766
			−−	−−	−−	−−

[1] $O.S.(\ell)$ denotes an optimal solution for SIP$\{Y_\ell\}$.

[2] $y^{(1)}_0 = 1$, $y^{(1)}_1 = 0.5$, $y^{(1)}_2 = 0.582382$, $y^{(1)}_3 = 0.589055$.

[3] $\mu^*_1 = \sum_{i=1}^{2} \lambda^{(1)}_i \delta_{x^{(1)}_i}$, where $\lambda^{(1)}_1 = 1.006391$, $\lambda^{(1)}_2 = 0.003832$, $x^{(1)}_1 = 0.022705$, $x^{(1)}_2 = 1$.

[4] $\mu^*_2 = \sum_{i=1}^{2} \lambda^{(2)}_i \delta_{x^{(2)}_i}$, where $\lambda^{(2)}_1 = 1.006365$, $\lambda^{(2)}_2 = 0.003478$, $x^{(2)}_1 = 0.022278$, $x^{(2)}_2 = 1$.

[5] $\mu^*_3 = \sum_{i=1}^{2} \lambda^{(3)}_i \delta_{x^{(3)}_i}$, where $\lambda^{(3)}_1 = 1.006383$, $\lambda^{(3)}_2 = 0.003487$, $x^{(3)}_1 = 0.022308$, $x^{(3)}_2 = 1$.

From Table 1, we see that $\mu^*_3 = \sum_{i=1}^{2} \lambda^{(3)}_i \delta_{x^{(1)}_i}$ is feasible and hence is also the optimal solution of LPM–$T^*_1(f^*)$.

It is obvious that $F(\text{LPM}) \subset F(\text{LPM–}T^*_1(f^*))$ and $\mu^*_3 \in F(\text{LPM})$, since

$$\inf_{y\in Y} \{\phi(\mu^*_3, y) g^*(y) - f^*(y)\} = 0.$$

("0" implies an accuration within six decimal points.) Therefore μ^*_3 is an optimal solution, and the optimal value is

$$V(\text{LPM}) = \int_X (1.5 - x)\, d\mu^*_3(x) = 1.488868.$$

EXAMPLE 4.2. *Consider the LPM with* $h(x) := 1.5 - x$, $\phi(x,y) := \cos(2\pi xy)$, *and* ν^*, ν *absolutely continuous with respect to m, having the density functions*

$$g^*(y) := 1, \quad \forall y \in Y,$$
$$f^*(y) := \sum_{n=1}^{\infty} \left(2 - \frac{1}{10^{n-1}}\right) \chi_{[\frac{1}{2n}, \frac{1}{2n-1}]}(y), \quad \forall y \in Y, \quad \text{respectively.}$$

That is, LPM is of the form below:

$$\begin{aligned}&\text{minimize} \quad \int_X (1.5-x)\,d\mu(x)\\&\text{subject to}\ \mu\in M^+(X), \quad \text{and}\\&\text{and} \int_X \cos(2\pi xy)\,d\mu(x) \geqslant \sum_{n=1}^{\infty}\left(2-\frac{1}{10^{n-1}}\right)\chi_{[\frac{1}{2n},\frac{1}{2n-1}]}(y), \quad \text{a.e. in } Y.\end{aligned}$$

As f^* is of bounded variation on Y and other data of this problem are the same as Example 4.1, we also find an optimal solution for LPM by virtue of the subproblems LPM–$T_k^*(f^*)$. Since f^* is monotone decreasing, we let $f^*=g_1-g_2$, where $g_1=0$ and $g_2=-f^*$, and we take $k=2$ in Algorithm (2) owing to $\max\{\|g_1\|_\infty, \|g_1\|_\infty\}=2$. As in Algorithms (2), we let

$$r_k := \inf_{y\in Y}\{\phi(\mu_k^*,y)g^*(y)-f^*(y)\},$$

and let $N_k\in N$ be such that $N_k\geqslant k$, $\inf_{y\in Y}[\phi(\mu_k^*,y)g^*(y)-T_{N_k}^*(f^*)(y)]\geqslant 0$, but $\inf_{y\in Y}[\phi(\mu_k^*,y)g^*(y)-T_{N_k+1}^*(f^*)(y)]<0$. The progress of Algorithm (2) is summarized in Table 2.

Table 2.

k	[1] $O.S.(k)$	[2] $V(k)$	r_k	N_k
2	[3] μ_2^*	2.695959	−0.011195	2
3	[4] μ_3^*	2.709825	0	−−

[1] $O.S.(k)$ denotes an optimal solution for LPM–$T_k^*(f^*)$.
[2] $V(k)$ denotes the optimal value for LPM–$T_k^*(f^*)$.
[3] $\mu_2^*=\sum_{i=1}^{2}\lambda_i^{(2)}\delta_{x_i^{(2)}}$, where $\lambda_1^{(2)}=1.836696$, $\lambda_2^{(2)}=0.186896$, $x_1^{(2)}=0.167725$, $x_2^{(2)}=0.167847$.
[4] $\mu_3^*=\sum_{i=1}^{3}\lambda_i^{(3)}\delta_{x_i^{(3)}}$, where $\lambda_1^{(3)}=1.743671$, $\lambda_2^{(3)}=0.285351$, $\lambda_3^{(3)}=0.014068$, $x_1^{(3)}=0.168823$, $x_2^{(3)}=0.168945$, $x_3^{(3)}=0.869263$.

Since $r_3=0$, by Algorithm (2) $\mu_3^*=\sum_{i=1}^{3}\lambda_i^{(3)}\delta_{x_i^{(3)}}$ is an optimal solution, and the optimal value is 2.709825.

EXAMPLE 4.3. *Consider the LPM with* $h(x) := 1.5 - x$, $\phi(x,y) := \cos(2\pi xy)$, *and* ν^*, ν *absolutely continuous with respect to m.* ν^* *and* ν *have density functions*

$$g^*(y) := e^{(y-1)}, \quad \forall y \in Y,$$
$$f^*(y) := \sum_{n=1}^{\infty} (2 - \frac{1}{10^{n-1}}) \chi_{[\frac{1}{2n}, \frac{1}{2n-1}]}(y), \quad \forall y \in Y, \quad \text{respectively.}$$

That is, LPM is of the form below:

$$\text{minimize} \quad \int_X (1.5 - x) d\mu(x)$$
$$\text{subject to } \mu \in M^+(X), \quad \text{and}$$
$$\text{and } e^{(y-1)} \int_X \cos(2\pi xy) d\mu(x) \geqslant \sum_{n=1}^{\infty} (2 - \frac{1}{10^{n-1}}) \chi_{[\frac{1}{2n}, \frac{1}{2n-1}]}(y), \quad \text{a.e. in } Y.$$

As in the examples above, we can solve this problem by Algorithm (2), since

$$\inf_{y \in Y} \{ e^{(y-1)} \int_X \cos(2\pi xy) d\delta_0(x) \} = \frac{1}{e} > 0.$$

Also we let $k = 2$ in the first step of Algorithm (2). The progress of Algorithm (2) is summarized in Table 3, where the same notation appears in Table 2.

Table 3.

k	$O.S.(k)$	$V(k)$	r_k	N_k
2	${}^1\mu_2^*$	6.057516	-10^{-1}	2
3	${}^2\mu_3^*$	6.236674	-10^{-2}	3
4	${}^3\mu_4^*$	6.254605	-10^{-3}	4
5	${}^4\mu_5^*$	6.256398	-10^{-4}	5
6	${}^5\mu_6^*$	6.256577	-10^{-5}	6
7	${}^6\mu_7^*$	6.256595	-10^{-6}	7

[1] $\mu_2^* = \sum_{i=1}^{3} \lambda_i^{(2)} \delta_{x_i^{(2)}}$, where $\lambda_1^{(2)} = 1.859129$, $x_1^{(2)} = 0.236328$, $\lambda_2^{(2)} = 2.635647$, $x_2^{(2)} = 0.23645$, $\lambda_3^{(2)} = 0.669959$, $x_3^{(2)} = 0.935913$.

2 $\mu_3^* = \sum_{i=1}^{3} \lambda_i^{(3)} \delta_{x_i^{(3)}}$, where $\lambda_1^{(3)} = 1.858127$, $x_1^{(3)} = 0.241211$,
$\lambda_2^{(3)} = 2.727346$, $x_2^{(3)} = 0.241699$, $\lambda_3^{(3)} = 0.823908$, $x_3^{(3)} = 0.934570$.

3 $\mu_4^* = \sum_{i=1}^{3} \lambda_i^{(4)} \delta_{x_i^{(4)}}$, where $\lambda_1^{(4)} = 2.786746$, $x_1^{(4)} = 0.241943$,
$\lambda_2^{(4)} = 1.807445$, $x_2^{(4)} = 0.242065$, $\lambda_3^{(4)} = 0.839655$, $x_3^{(4)} = 0.934204$.

4 $\mu_5^* = \sum_{i=1}^{3} \lambda_i^{(5)} \delta_{x_i^{(5)}}$, where $\lambda_1^{(5)} = 0.905336$, $x_1^{(5)} = 0.241943$,
$\lambda_2^{(5)} = 3.689778$, $x_2^{(5)} = 0.242065$, $\lambda_3^{(5)} = 0.841178$, $x_3^{(5)} = 0.934204$.

5 $\mu_6^* = \sum_{i=1}^{3} \lambda_i^{(6)} \delta_{x_i^{(6)}}$, where $\lambda_1^{(6)} = 0.717194$, $x_1^{(6)} = 0.241943$,
$\lambda_2^{(6)} = 3.878012$, $x_2^{(6)} = 0.242065$, $\lambda_3^{(6)} = 0.841330$, $x_3^{(6)} = 0.934204$.

6 $\mu_7^* = \sum_{i=1}^{3} \lambda_i^{(7)} \delta_{x_i^{(7)}}$, where $\lambda_1^{(7)} = 0.698380$, $x_1^{(7)} = 0.241943$,
$\lambda_2^{(7)} = 3.896835$, $x_2^{(7)} = 0.242065$, $\lambda_3^{(7)} = 0.841346$, $x_3^{(7)} = 0.934204$.

Since $r_7 < 0$, μ_7^* is not feasible for LPM and hence it is not an optimal solution. However, by the same argument as Theorem 3.7, we have

$$\begin{aligned} 0 &\leqslant V(\mathrm{LPM}) - V(\mathrm{LPM}\text{–}T_7^*(f^*)) \\ &\leqslant \frac{\sup_{y\in Y}\{f^*(y) - T_7^*(f^*)(y)\}}{\inf_{y\in Y} \phi(\delta_0, y) g^*(y)} \int_X (\frac{3}{2} - x)\, d\delta_0(x) \\ &= \frac{3}{2} e \cdot 10^{-6}. \end{aligned}$$

Hence, the approximate value $V(7) = 6.256595$ is correct to five decimal places.

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